Investigating the Teaching and Learning of Negative Number Concepts and Operations

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A report of progress in a Doctor of Education research project supervised by Professor Kaye Stacey.

The paper provides an overview of some of the outcomes of an investigation comparing the teaching of negative number at junior secondary level using tiles, as discrete integer entities, with other teaching approaches more commonly used. The experimental approach seems to have facilitated better performance for average ability level students. For more able mathematics students the topic does not appear to be difficult and such students, in both experimental and control groups, indicated good levels of general topic mastery.

Introduction and background

Much has been written and critical comment made concerning suitable and effective negative number teaching strategies (eg. Fary (1980), Kuchemann (1981), Freudenthal (1983), Streefland (1993), White (1994), Lytle (1994), Gates (1995)). It continues to be one of the ongoing sagas in mathematics education. However improved learning and understanding for many students is still not being achieved. Recently, at the beginning of a workshop on the teaching of negative number, not one of the twelve fourth year trainee mathematics teachers present could prove or give an explanation for either '0 - -6 = 6' or '-2 x -3 = 6'. They appeared to merely know or accept them to be true. None could recall how (or if) such facts had been taught or justified. A large proportion of students emerge from secondary school with a seriously flawed and incomplete understanding of the real number system. Any area or application of mathematics requiring the use of negative numbers and related concepts is likely to produce difficulties. Ongoing development in mathematics and mathematics related fields is therefore adversely affected.

Two years ago at MERGA in Lismore a paper was presented describing the, then beginning, project to study the teaching and learning of negative number concepts and operations (Hayes, 1994). Also outlined were the experimental teaching methods to be investigated. This paper will provide details of what has occurred since and offer some preliminary findings. In particular it discusses some of the outcomes of school-based teaching and testing activities that took place in 1995 and early 1996. The data analysis process has not yet been completed. It is hoped that the project will be completed by the end of this year.

Experimental Details

In the period mid-1994 to term one 1996, the study has been conducted in three secondary schools involving students in years seven, eight and nine, comparing the effectiveness of the annihilation/creation method of teaching negative number (see Freudenthal, 1983) as the initial strategy, with other commonly used strategies. The contention is that the method provides a superior introduction and operational model for the teaching of integer operations and will result in improved long term learning and understanding, particularly for students who have difficulty in mathematics. The experimental teaching groups used reversible two centimetre square tiles labelled [+1], [-1] and [0] in conjunction with especially prepared self-paced student workbooks covering the four basic integer operations. It was intended that the topic would fit into the normal time devoted to it in the syllabus (about three weeks) at either late year seven or early year eight depending on usual school policy. An aim of using the tiles in conjunction with the workbooks was for the students to discover and formulate for themselves the integer operation rules. Teaching methods in comparison groups were those traditionally used by the teachers of the control classes involved in the study. In general number-line interpretations and applications tended to feature strongly as the preferred initial teaching strategy used. Consistency patterns were commonly used to justify (establish) the multiplication rules. The major difference in strategy between the experimental and control groups was that the experimental groups started with the tiles. By the end of the topic the experimental group students had also used the number line in the context of ordering and 2D point plotting. Across the three schools there were four experimental groups and four control/comparison groups involving a total of 150 students and seven teachers. One of the teachers took both an experimental group and a control group.

Participants

Three schools were involved. School A provided two year 8 classes one of which was the experimental group and the other the control group. Both were taught by the same teacher. The experimental group was allowed to work in self-paced mode, using the materials provided (tiles and workbooks) throughout the three weeks devoted to the topic early in term one. Students were allowed to take the workbooks away for the purpose of homework. The control group was taught by the usual class-lesson methods favoured and described by the teacher as a multi-embodiment approach. The number-line was used as the introductory strategy for teaching the processes of integer addition and subtraction. The textbook used in the control group classes was *Mathematics Today Year 8*.

School B provided two year 8 classes and the teaching also took place early in term one. In this case different teachers taught the experimental and control groups. The experimental group worked with the tiles and workbooks provided in self-paced fashion. Workbooks were collected for correction following each lesson and not taken away by the students for homework In this case the class teacher appeared to provide the students with more detailed individual written feedback comments and corrections than those given by the teacher in School A above. The control group teacher appeared to generally follow the topic treatment adopted by the class textbook used (*Mathematics Today, Year 8*). In the latter half of 1995 one of the teachers involved (who was also the mathematics coordinator) was transferred to another school. This caused some problems and organisational disruptions which affected access to the control group for the purpose of retention testing later in the year.

School C initially provided three year 7 classes. The teaching took place in term three, 1995. Initially there were two experimental groups and one control group. Each class was taken by a different teacher. In this school the experimental group teachers monitored student progress very closely and tended to keep the classes more together (ie. a minimum pace was set) but students were also allowed to take their workbooks home and work ahead. Class reviewing and discussion of the content material was most apparent. One of the experimental group teachers regularly used a process of page by page class corrections for some sets of workbook examples. (eg. "David, please read out your answers for the exercises on page 10.") The control group teacher used the number line approach and textbook exercises from Lynch et al. *Maths 8,* supplemented by additional prepared review worksheets. Retention-testing was done in February, 1996. Because classes had been regrouped for the transition to year 8 it was decided to minimize class disruption by retention-testing all year 8 students. This included additional students (an accelerated group) to those who had previously been involved in the study in 1995 - effectively providing an extra control (comparison) group.

Testing

Students were pre-tested (usually at the start of the first lesson in the topic), post-tested (usually the next maths period following the topic completion) and then retention-tested (four to six months later). In 1995 the pre-test covered basic positive number facts in the form of addition and multiplication scramble tables and simple positive and negative number operation examples (eg. $3 + {}^{-}3$, $5 + {}^{-}2$, ${}^{-}3 + {}^{-}2$, ${}^{-}4 \times {}^{-}2$ etc.). The scramble tables were intended as indicators of readiness to begin the topic. (None of the students were found to be grossly unready - all had at least reasonable knowledge of basic simple number facts.) The integer operation examples were intended to provide an indication of naturally acquired knowledge and feeling for the topic. At that stage it was expected that few, if any, of the students would have experienced formal teaching in the topic. However most students, at this level, seem to be aware of at least one example of practical usage (ie. temperature scale). The pre-test showed that most students seemed to know that $3 + {}^{-}3 = 0$ and ${}^{-}3 + {}^{-}2 = {}^{-}5$. Many students performed quite well on the simpler examples included (eg. $7 + {}^{-}4$) but most did not know (or correctly guess) that ${}^{-}4 \times {}^{-}2 = 8$ (the common response was ${}^{-}8$). Five students (one in each of Schools A and C initial groups) achieved full marks (14/14) on the integer operations part of the pre-test.

Both the post-test and the retention-test included one common large question (Q1.) containing 30 items covering the four processes, using small integers and intended to test knowledge of and competence in basic operations. The numerals in the items were changed between the tests. The post-test also covered some additional skills and integer applications traditional at this level (eg. ordering, temperature scale, above and below sea-level questions). Apart from Q1. the content of the post-test was modified for School C. The intention was to make comparisons within schools rather than across schools at the post-test stage. However the retention-test used was common to all 1995 participants. In the retention-test Q2. contained 20 items testing knowledge of the operation rules. Using large numbers the task involved deciding whether the answer in each case would be positive (P), negative (N) or zero (Z) for each of the binary operation examples. The 12 items in Q3. involved substituting small integers into simple algebraic expressions and then evaluating each.

Early in 1996 a selection of 30 students of varying ability, involved in the study in 1995, were interviewed and asked to think aloud and provide explanations and reasons whilst they responded to a selection of items similar to those contained in questions 1 and 2 of the retention-test. One of the intentions was to unravel reasons for wrong answers. Working and voices, but not faces, were video-recorded. (At the time of writing this paper the data (video tapes and response sheets) from the interviews were still being analysed.)

Results and discussion

To facilitate data analysis and comparisons for parallel groups in each school the responses of all students to all items on the tests were entered into EXCEL spreadsheets. (Tables 2, 3 and 4 which are discussed later show sample extracts.) Due to the diverse conditions existing between the three schools involved (eg. nature of student intake, staffing stability, policy) only a minor attempt is being made to directly compare student performances between schools. For the purpose of this study each school has been regarded as somewhat of a separate case study and major comparisons made within schools rather than across schools. In fact, as this study proceeds, it is becoming quite evident that the reactions of the individual students appear to provide the most interesting features and indicators of teaching strategy effectiveness. Detailed analysis of workbooks is also still in progress.

Apart from the School C accelerated group, who emerged, as mentioned above, at the retention-testing stage, each of the schools regarded the respective parallel groups participating in the study as containing generally similar ability ranges. It would not have been possible to randomly allocate students to groups without causing major disruptions in any of the schools.

Table 1. provides mean scores and standard deviations for the testing program for the 1995 teaching groups. (Ttests have been used to provide possible indicators of within school group differences.)

Table 1. Test results for students in the 1995 teaching groups

TEST RESULTS								
School	**Group**	**Pre-test**	**Post-test**		**Retention-test**			
		Basic op.	Basic op.	Ov%score	Basic op.	Rules	Substitns.	Ov%score
		/14	/30		/30	/20	/12	
A	Exptl.	9.3(m)	22.0	73.8	21.8	16.1	6.7	72.0
Year 8	N=22	3.0(sd)	5.6	16.0	6.5	3.0	3.7	19.7
Term 1	Control.	6.9	18.4	60.7	17.2	13.2	5.5	57.8
	N=22	4.4	8.2	23.2	7.2	3.3	3.3	20.1
		p<.05	ns	p<.05	p<.05	p<.05	ns	p<.05
B	Exptl.	5.1	18.5	62.5	23.3	13.4	6.7	70.1
Year 8	N=17	3.7	7.7	21.1	7.4	7.9	4.9	20.1
Term 1	Control.	8.5	18.9	60.4				
	N=19	3.5	7.7	27.1				
		p<.05	ns	ns				
C	Exptl.1	8.4	25.6	74.2	26.8	18.0	10.1	88.4
Year 7	N=17	2.1	4.3	17.3	3.3	2.4	1.4	11.4
Term 3	Exptl.2	8.7	26.0	75.6	26.4	17.4	10.0	86.4
	N=18	2.6	4.4	19.0	5.8	3.8	2.0	17.9
	Contrl.1	6.4	24.6	65.7	25.9	17.2	8.6	83.5
	N=13	2.7	5.5	21.0	4.5	2.5	3.0	14.3
	Contrl.2				26.9	17.8	9.4	87.2
	N=22				5.4	2.9	2.4	16.7
		p<.05	ns	p<.05	ns	ns	ns	ns

School A (Both classes taught by the same teacher): Prior to commencement of teaching the topic, pre-test scores, (mean 9.3) obtained on the 14 basic integer items by the experimental group, appeared to indicate a reasonably good intuitive idea for the correct answers. This was significantly higher (Ttest, p<.05) than the control group mean (6.9). However whilst all members of the experimental group attempted (perhaps sometimes guessed) these items four of the control group did not attempt any and two others attempted only a few. (Four zero and two other very low scores considerably reduced the class mean.) One such student included a note; "I haven't done these yet and I don't know them." Another provided the comment, "I don't know what the dashes in front of the numbers mean." The experimental group performed significantly better on the post and retention-tests. Overall post-test mean scores were 73.8% and 60.7% respectively (p<.05) and the overall retention-test scores 72.0% and 57.8% (p<.05). Before the teaching commenced the teacher, whilst deciding which would be the experimental and control group, considered the classes to be of about the same ability levels and, following teaching, predicted that the control group would have performed better on the post-test than the experimental group when asked, "Which group do you think did better?". The teacher was surprised by the difference in measured performance levels between the two groups on the post-test.

School B (Classes taught by different teachers): In this case the control group (mean 8.5) scored better than the experimental group (mean 5.1) ($p<.05$) on the pre-test integer items. However the gap had closed by the post-test and the overall mean score (62.5%) by the experimental group was slightly higher than that of the control group (60.4%) (ns). Unfortunately the retention-test was not administered to the control group by the school. (It is hoped that both groups can be re-tested simultaneously again soon.)

School C (All classes taught by different teachers): On the pre-test the experimental groups performed better than the initial control group. The overall post-test mean scores of the experimental groups (74.2% and 75.6%) were significantly better than those achieved by the initial control group (65.7%). However mean scores on the 30 basic operations items were similar; 25.6, 26.0 and 24.6 respectively. On the retention-test the initial control group scored slightly lower (ns) than the other three groups in each of the aspects of the test however the overall results indicate a high level of general mastery in the topic for each of the classes involved. The two experimental groups performed as well

Table 2. Item analysis retention-test spreadsheet extract (School A experimental group)

Q1. .	q	r	s	t	u	v	w	x
Content .	-27÷[ ]=-9	2 x (7-8)	-3 x (2--4)	-15 ÷(1-4)	0 - -2	-4 -[ ]= -8	-3 x 6	-4 x -2
Corr.Resp	3	-2	-18	5	2	4	-18	8
Annie	1	2	-1	3	-2	*	18	1
Bert	1	-1	6	1	-2	1	1	1
Claude	1	1	6	1	1	12	1	1
Dot	1	2	*	*	-2	-4	-9	-8
Eddie	1	1	1	1	0	-12	1	1
Fay	1	2	-6	-5	-2	-4	1	-8
Gladys	1	1	18	1	-2	1	18	1
Harry	1	2	18	-5	-2	-4	18	1
Irma	1	1	18	1	-2	1	1	1
Jay	-3	1	18	1	1	1	18	1
Kiran	1	1	1	1	1	1	1	1
Les	1	2	1	1	1	1	1	1
Mavis	1	2	5	-5	-2	-4	1	1
Norm	x	2	1	1	1	x	1	1
Ossie	1	1	1	1	1	1	1	1
Pete	1	1	+6	-5	-2	1	1	1
Quaid	1	1	1	1	1	1	1	1
Rex	1	1	1	1	1	1	1	1
Sam	1	1	18	-5	1	1	1	1
Thelma	1	1	18	-5	1	1	1	1
Una	1	1	1	1	1	1	1	1
Viv	1	1	1	1	1	1	1	1
#correct	20	14	9	14	12	14	17	19
#wrong	2	8	13	8	10	8	5	3
%correct	90.9	63.6	40.9	63.6	54.5	63.6	77.3	86.4
%wrong	9.1	36.4	59.1	36.4	45.5	36.4	22.7	13.6

as the accelerated group who had been selected to work above their level in mathematics and other subjects. In mathematics it appears that they covered the same syllabus as the other control class but at a faster pace and therefore covered integers earlier in the year. (They worked from Lynch et al. *Maths 8* during most of Year 7.)

Analysis of error patterns

Interesting data has emerged with regard to error patterns among students who made mistakes. In addition to experimental and control group comparisons the spreadsheets provide useful instruments for diagnostic purposes. It is suggested that such information could facilitate the selection of teaching strategies for overcoming and avoiding misunderstanding. Table 2 and Table 3 show selected and edited extracts from the School A experimental group and control group retention-test spreadsheets respectively. The tables show eight out of the overall total of 62 items contained in the test. The '1' on the right hand side of cells indicate that the students gave correct item responses. The numbers shown on the left hand side of cells indicate students' incorrect item responses. Asterisks (*) indicate non-attempts. Several non-attempts are apparent in the control group whilst none appear for the selected items for the experimental group. At the foot of the spreadsheets the item statistics for the class are obtained. The spreadsheets thus provide a rapid method of systematically analysing the class results. A major useful

Table3. Item analysis retention-test spreadsheet extract (School A control group)

Q1. .	q	r	s	t	u	v	w	x
Content .	-27÷[ ]=-9	2 x (7-8)	-3 x (2--4)	-15 ÷(1-4)	0 - -2	-4 -[ ]= -8	-3 x 6	-4 x -2
Corr.Resp	3	-2	-18	5	2	4	-18	8
Alf	-3	*	*	*	*	*	*	*
Bess	1	1	-6	1	-2	*	*	*
Connie	1	2	1	1	1	1	1	1
Dave	1	1	-12	15	1	1	1	-8
Edna	-3	2	-9	-5	-2	1	-9	1
Fred	*	*	*	*	*	*	*	*
Gough	-3	1	-6	-5	-2	1	1	-8
Helen	-3	1	18	1	1	1	1	1
Ivan	1	1	-12	15	-2	1	1	-8
Jill	*	1	*	*	*	*	*	*
Keith	1	1	1	1	1	1	1	1
Lora	1	1	1	1	1	-4	1	-8
Mick	1	1	18	-5	1	1	1	1
Ness	*	*	*	*	*	*	*	*
Olga	1	1	1	-5	1	1	1	*
Pam	-3	1	1	-5	1	1	1	1
Quinn	1	1	18	1	-2	1	1	1
Ray	1	1	*	*	*	*	*	*
Sally	1	1	1	1	1	1	1	1
Tommy	1	1	9	1	-2	*	1	1
Uno	1	1	6	1	0	1	18	1
Vida	-3	2	-9	-5	-2	1	-9	1
#correct	13	15	6	9	9	14	13	11
#wrong	9	7	16	13	13	8	9	11
%correct	59.1	68.2	27.3	40.9	40.9	63.6	59.1	50.0
%wrong	40.9	31.8	72.7	59.1	59.1	36.4	40.9	50.0

diagnostic feature is the range of incorrect responses for each item and the opportunity to analyse students' thinking. Item 1s, '-3 x (2 - -4)' was by far the most difficult (among this selection) for both groups. Several students in both groups provided '18'. These students have apparently evaluated the bracketed part incorrectly as -6 and then given -3 x -6 correctly as 18. In item 1r, '2 x (7-8)', the control group performed slightly better than the experimental group and for item v, '-4 - [ ] = -8', the same proportion of students in each group gave the correct response. Note, however, that seven control group students did not respond. For each of the other selected items the experimental group produced more correct responses. The 'x' shown for 'Norm' (experimental group) in both items q and v were actual responses (" 'x' is the unknown number?").

Across both the post and retention-tests the experimental group performed better than the control group on a large majority of items in the tests. On the retention-test the experimental group performed better on 56 out of the total of 62 items in the test, both groups provided the same proportion of correct answers on four of the items and the control group did better on only two items (Item 1r mentioned above and item 31, 'evaluate ab ÷ c if a = 4, b = -3 and c = -2').

Table 4 shows the same portion of retention-test result spreadsheet for one of the School C experimental groups. A high level of mastery is apparent. The few error responses made are similar to some of those made by students shown in Table 2 and Table 3 above.

Table 4. Item analysis retention-test spreadsheet extract (School C experimental group)

Q1. .	q	r	s	t	u	v	w	x
Content .	-27÷[ ]=-9	2 x (7-8)	-3 x (2--4)	-15 ÷(1-4)	0 - -2	-4 -[ ]= -8	-3 x 6	-4 x -2
Corr.Resp	3	-2	-18	5	2	4	-18	8
Alice	1	1	1	1	1	1	1	1
Bill	1	1	1	1	1	1	1	1
Connie	1	1	1	1	1	1	1	1
Daisy	1	1	1	1	1	1	1	1
Elsie	1	1	1	1	1	12	+18	1
Foster	1	1	1	1	-2	1	1	1
Gill	1	1	1	1	1	1	1	1
Harry	1	1	-6	1	1	1	1	1
Ivan	1	1	1	1	1	1	1	1
Jack	1	1	1	1	1	1	1	1
Kylie	1	1	1	1	1	1	1	1
Lucy	1	1	18	1	1	1	1	1
Maisy	1	1	1	1	1	1	1	1
Nellie	1	1	1	1	1	12	1	1
Olive	1	1	1	1	1	-4	1	1
Peter	1	1	1	1	1	1	1	1
Quok	1	1	-12	1	1	1	1	1
#correct	17	17	14	17	16	14	16	17
#wrong	0	0	3	0	1	3	1	0
%correct	100.0	100.0	82.4	100.0	94.1	82.4	94.1	100.0
%wrong	0.0	0.0	17.6	0.0	5.9	17.6	5.9	0.0

Conclusions

The following preliminary findings have emerged at this stage.

- For students who are performing well in mathematics, the method of initially teaching the topic of negative numbers does not seem to affect their long-term performance. In general the topic is not difficult for such students. More able students in both experimental and control groups displayed similar levels of mastery on basic operations, knowledge of the rules and algebraic substitutions and evaluations.

- Lower ability experimental groups appeared to achieve better scores than similar ability control groups. There are some indications that the tile method may be a superior initial teaching method for average and below average students. A major strength and characteristic of the tile method of teaching the topic is the easy and natural way in which it models integer operations and facilitates classroom and small group discussion and interaction.
- By the end of the topic both experimental and control group students tended to perform required operations automatically using the rules. Only a few students used diagrams (eg. tile representations or number lines) when doing the tests. (As the topic progressed some experimental group students tended to draw small tile-like representations rather than use the actual tiles to do some exercises.)
- It was not possible to control for teacher effect and student attitude. Both factors may have affected outcomes. The influence of teachers (eg. control, provision of corrections and feedback, systematic monitoring of progress) and the attitudes of students (eg. reflected in the rate and thoroughness in which they tackled assigned tasks) appears to have been particularly evident in School C. Having the same teacher in charge of both groups in School A should have helped control for teacher effect, however there may have been attitude problems affecting test performance for some students. In both School A and School B there may have been a degree of test underachievement due to lack of effort by some students because some tests (pre and retention) didn't count toward assessment. Written comments on a few test papers included; "This doesn't count" and " I don't have to do this". This is an argument for attempting teaching and learning studies, on topics such as this, in the context of normal school classroom programs. All students involved, because they were required to give signed consent - an ethics committee condition, knew that this was part of an experiment.

References

Daly, T. et al. (1993) *Mathematics Today Year 8* (2nd Edn.) Sydney: McGraw-Hill.

Fary, B. (1980) Children's Use of Imagery in the Learning of a Mathematical Skill. Unpublished M.Ed. Thesis, Monash University, Clayton, Victoria.

Freudenthal, H. (1983) *Didactical Phenomenology of Mathematical Structures*, Dordrecht: Reidel.

Gates, L. (1995) Product of two negative numbers: An example of how rote learning a strategy is synonymous with learning the concept. *Proceedings of the 18th Annual Conference of the Mathematics Education Research Group of Australasia*, Darwin, Northern Territory, p.278-284.

Hayes, B. (1994) Becoming more positive with negatives, *Proceedings of the 17th Annual Conference of the Mathematics Education Research of Australasia*, Lismore, NSW, p.331-338.

Kuchemann, D. (1981) Positive and Negative Numbers in Hart, K. (ed.) *Children's Understanding of Mathematics: 11-16*, London: John Murray.

Lynch, B. J., Parr, R. E., Picking, L. P. & Keating, H. M. (1991) *Maths 8*, (2nd Edn.) Melbourne: Longman Sorret.

Lytle, P. (1994) Investigation of a model based on the neutralization of opposites to teach integer addition and subtraction, *Proceedings of the 18th International Conference for the Psychology of Mathematics Education*, Lisbon, Portugal, Vol.3, p.192-199.

Streefland, L. (1993) Negative numbers: concrete & formal in conflict?, *Proceedings of the 16th Annual Conference of the Mathematics Education Research Group of Australasia*, Brisbane, Queensland, p.531-536.

White, A. (1994) Hurdles, who put them on the directed number line?, *The Australian Mathematics Teacher*, 50 (1), p.14-17.